INTRODUCTION TO PREDICATE LOGIC
NOTES

Chapter 1.

In predicate logic we study the logical form INSIDE atomic (shortest) sentences. To say: "Adam is blond." is to attribute the property of being blond to Adam. So let us use the capital letter 'B' to stand for the predicate, 'is bond', and the lower case letter 'a' to stand for the name 'Adam'. We say, "Adam is blond" by writing: 'Ba'.

To say: "Adam loves Eve." is to say that the relation of loving ('L') holds between Adam ('a') and Eve ('e'). In predicate logic we say "Adam loves Eve." by writing 'Lae'.

Sentences such as 'Ba' and 'Lae' count as atomic (shortest) predicate logic sentences. They can be combined with the &, v, -, ->, and <-> just as atomic sentence letter (which we can still use). Thus to say that Adam loves Eve if she is blond we write: 'Be -> Lae'.

Exercises: In the following exercises, use this transcription guide:

a: Adam
e: Eve
c: Cid
Bx: x is blond
Cx: x is a cat
Lxy: x loves y
Txy: x is taller than y

1-1 a) Tce, b)Lce, c) -Tcc, d) Bc, e) Tce -> Lce,
f) Lce v Lcc, g) -(Lce & Lca), h) Bc <-> (Lce v Lcc)

1-2. a) Cid is a cat. b) Cid is taller than Adam. c) Either Cid is a cat or he is taller than Adam. d) If Cid is taller than Eve then he loves her. e) Cid loves Eve if he is taller than she is. f) Eve loves both Adam and Cid. g) Eve loves either Adam or Cid. h) Either Adam loves Eve or Eve loves Adam, but both love Cid. i) Only if Cid is a cat does Eve love him. j) Eve is taller than but does not love Cid.

In English we can say, not only that Eve loves Adam, but that someone loves Adam, or that everyone loves Adam. We can also use pronouns, 'he', 'she', and 'it' to cross reference: For example, I can say that if someone loves Adam, then he or she is blond. In predicate logic VARIABLES, w, x, y, and z function much like English pronouns. And we use these variables in new symbols, called QUANTIFIERS, to get the force of words like 'someone', 'everybody', 'something' and 'everything'. Read the predicate logic sentence, '(x)Bx' as 'Every x is such that x is blond.', or more simply as , 'Everything is blond'. Note that in English there is a difference between 'everything' and 'everyone' (or 'everybody'). In more advanced work we can make this distinction in predicate logic also, but while learning basic ideas we will ignore this distinction in logic: Transcribe '(x)' as 'everything' or as 'everyone' (or the equivalent 'everybody') as seems appropriate in the context.

The same kind of remarks go for 'some': 'Somebody is blond' gets transcribed as '(Ex)Bx', read in a literal minded way as "There is an x such that x is blond." Again, we ignore the difference between something' and 'someone' (or 'somebody')

'(x)' is called the UNIVERSAL QUANTIFIER. (Ex) is called the EXISTENTIAL QUANTIFIER. The quantifiers can also use the other variables. We transcribe 'Someone is blond' equally well as '(Ex)Bx' and '(Ey)By'.

Quantifiers can apply to compound as well as to atomic sentences. Thus we say that someone is blond and in love with Adam by writing: '(Ex)(Bx & Lxe)'. And to say that everyone either loves Adam or loves Eve we write (x)(Lxa v Lxe)'. The application of a quantifier is in this respect similar to the application of the negation sign: A quantifier applies to the shortest full sentence which follows it, as indicated by parentheses.

As in discussing sentence logic, we use boldface capital **X** and **Y** to stand for arbitrary sentences. Similarly, we use bold lower case **u** and **v** to stand for arbitrary variables. [We also will use boldface 's' and 't' to talk generally about names.] The following summarizes the new ideas in this section:

> We use lower case 'a' through 'r' as names and 'w', 'x', 'y', and 'z' as variables. Capital letter 'A' through 'W' can be used as sentence letters (when followed by no names or variables), as predicates (when followed by one name or variable), and as relation symbols (when followed by two or more names and/or variables). Any of these count as the atomic sentences of predicate logic. Any sentences of predicate logic can be combined with the &, v, -, ->, and <-> to form compound sentences. Finally, if **X** is a sentence of predicate logic, then so are (**u**)**X** and (E**u**)**X**

Exercises 1-4: Using the same transcription guide as in the previous exercises, transcribe the following into English:

a)-Laa, b) Laa -> -Taa, c) -(Bc v Lce), d) Ca <-> (Ba v Lae), e) (Ex)Txc, f) (x)Lax & (x)Lcx, g) (x)(Lax & Lcx), h)(Ex)Txa v (Ex)Txc, i) (Ex)(Txa v Txc), j) (x)(Cx -> Lxe), k) (Ex)(Cx & -Lex), l) -(x)(Cx -> Lex), m) (x)[Cx -> (Lcx v Lex)], n) (Ex)[Cx & (Bx & Txc)]

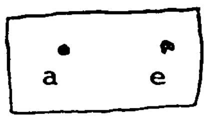
1-5: Transcribe the following into sentences of predicate logic. a) Everyone loves Eve. b) Everyone is loved by either Cid or Adam. c) Either everyone is loved by Adam or everyone is loved by Cid. d) Someone is taller than both Adam and Cid. e) Someone is taller than Adam and someone is taller than Cid. f) Eve loves all cats. g) All cats love Eve. h) Eve loves some cats. i) Eve loves no cats. j) Anyone who loves Eve is not a cat. k) No one who loves Eve is a cat. l) Somebody who loves Adam loves Cid. m) No one loves both Adam and Cid.

Chapter 2.

In sentence logic we gave the meaning of each connective with a truth table definition, which tells us, for each possible case, whether a sentence formed with the connective is true or false in that case. We do the same kind of thing to give a precise description of the meaning of the quantifiers. But now our characterization of a "possible case" is more complicated than a line of a truth table.

A possible case in which a predicate logic sentence will be true or false is callsed an INTERPRETATION. An interpretation consists of a collection of things and then a specification of what predicates are true of those things and which relations hold between those things, for all relevant predicates and relations. For example,

Ba,	Be,	Laa,	Lae,	Lea,	Lee,
t	f	f	t	f	t



Imagine that we are describing a very simple universe with only two things in it, called 'a', and 'e', and you are writing a novel about a and e. The information to the left of the box gives the atomic facts true in your novel, or possible case. Other possible cases, or interpretations, will have other things in them and different specification about what is true and what is false about these things.

An existentially quantified sentence is true in an intepretation just in case it is true for AT LEAST ONE thing in the interpretaion. In our example just given, '(Ex)Bx' is true and '(Ex)Lxa' is false. Be sure you understand why. A universally quantified sentence is true in an interpretation just in case it is true for ALL things in the interpretation. In our example, '(x)Lxe' is true and '(x)Bx' is false. Again, be sure you understand why.

The following definitions make all of these ideas precise:

An INTEPRETATION consists of

a) A DOMAIN, consisting of the objects in the interpretation, always including at least one object.
b) One, or more, names for each object
c) A list of zero place predicates (that is sentence letters), one place predicates (such as 'is blond'), and two or more place predicates (what I have been calling relation symbols).
d) Truth values for all atomic sentences which can be formed from the names and (zero place, one place, and many place) predicates.

An INTERPRETATION OF A SENTENCE is an interpretation including at least all the predicates and names occurring in the sentence.

(Incomplete definition) A SUBSTITUTION INSTANCE of a universally quantified sentence (u)(...u...), with the name s substituted for u, is (...s...), the result of dropping the '(u)' and substituting 's' for 'u' in the rest of the sentence. A SUBSTITUTION INSTANCE of an existentially quantified sentence (Eu)(...u...), with the name s substituted for u, is (...s...), the result of dropping the '(Eu)' and substituting 's' for 'u' in the rest of the sentence.

(Incomplete definition) A UNIVERSALLY QUANTIFIED SENTENCE IS TRUE IN AN INTERPRETATION just in case ALL substitution instances which you can form using names in the interpretation are true in the interpretation. An EXISTENTIALLY QUANTIFIED SENTENCE IS TRUE IN AN INTERPRETATION just in case ONE OR MORE substitution instance which you can form using names in the interpretation are true in the interpretation.

Exercises: 2-1 Make up an interpretation for each of the following sentences. Present the domain of your interpreations by writing D = { , , ...}, with the names writen between the commas and present the predicate letters and truth values much as we listed counterexamples for truth trees. For example, one interpretation for the sentence 'Tb & Kbd' is:

D = {b,d}; Tb & Td & Kbb & Kbd & Kdb & Kdd.

Remember, in this problem you are COMPLETELY FREE to make up any interpretation you want as long as it involves all the names and predicates appearing in the sentence. a) Lab, c) Lab v -Lba e) Ga & (Ex)(Lxb v Rax).

2-2. Consider the interpretation

D = {a,b}; -Ba & Bb & Laa & -Lab & Lba & -Lbb.

Give all of the substitution instances in this interpretation, for each of the following sentences, and say of each of the substitution instances whether it is true or false in the intepretation.

a) (Ex)Bx, c) (x)Lxa, e) (x)(Bx v Lax), f) (Ex)(Lxa & Lbx), g) (x)(Bx -> Lbx) h) (Ex)[(Lbx & Bb) v Bx]

2-3. For each of the sentences in exericse 2-2 give the truth value of the sentences in the given interpretation.

2-4. Determine the truth value of each of the following sentences in the interpretation of exericse 2-2. Be careful to determine the main connective and remebmer that a quantifier is a connective which applies to the shortest full sentences which follows it. Always apply the rule for truth values which applies to the main connective.

a) (Ex)Lxx -> (x)(Bx v Lbx), b) -(Ex)(Lxx -> Bx)&(x)(Bx -> Lxx)
c) (Ex)[Bx <-> (Lxa v Lxb)],
e) -(x)(-Lxx v Lxb) -> (Lab v -Lba),
g) (x)-[(-Lxx <-> Bx) -> (Lax <-> Lxa)]

The idea of validity for predicate logic is the same as it was for sentence logic: An argument is valid if and only if all possible cases in which all the premises are true are also cases in which the conclusion is true. The only change is that we now have a more general idea of what counts as the possible cases, namely interpretations. Consequently:

> A predicate logic argument is VALID if and only if every interpretation in which all the premises are true is one in which the conclusion is true.

The idea is equivalently expressed with the idea of a counterexample:

> A COUNTEREXAMPLE to an argument with predicate logic sentences is an interpretation in which all the premises are true and the conclusion is fale.

> A predicate logic argument is VALID just in case there are no counterexamples to it.

Exercises 2-6. For each of the following arguments, determine whether the argument is valid or invalid. If invalid, show this by presenting a counterexample. You will have to concoct counterexamples by paying careful attention to what the sentences mean. If there appear to be no counterexamples, try to explain informally why there are none, which will provide informal reasons for taking the argument to be valid. For example, consider the argument: (x)Lxe/Lae (Everyone loves Eve. Therefore Adam loves Eve). Suppose we had a counterexample, that is an interpretation in which the premise is true and the conclusion false. But for the premise, a universally quantified sentence, to be true in an interpretation, all its substitution instances have to be true in the interpretation. And the conclusion is just one of these substitution instances.

a)	(x)Lxe	b)	Lae	c)	(Ex)Lxe	d)	(x)(Bx & Lxe)
	------		----		-------		------------
	(Ex)Lxe		(x)Lxe		Lae		(x)Bx

Chapter 3.

So far I have avoided the complication of having one quantifier piled on top of another. Compare '(x)(Ey)Lxy' ('Everyone loves someone') with '(Ex)(y)Lxy' ('Someone loves everyone'). These clearly make very different statements. Not only the order of the quantifiers, but also the order of the variables can make a big difference. Compare the last two sentences with '(x)(Ey)Lyx' and '(Ex)(y)Lyx'

All this requires a little care when the same variable gets used in two different quantifiers. How would you understand '(Ex)[(x)Lxa -> Lxb]'? The key here is to remember how the sentence gets built up from its parts. Starting with the atomic sentence 'Lxa' we prefix '(x)' to get '(x)Lxa'. In this sentence the 'x' in 'Lxa' is BOUND by the '(x)'. Next we form a conditional with '(x)Lxa' and 'Lxb' to get '(x)Lxa -> Lxb'. In this sentence the last occurrence of 'x' is FREE. This is because the '(x)' applies only to the shortest full following sentence, the sentence in its SCOPE, binding all free occurrences of 'x' in its scope. Finally, we prefix the whole sentence with '(Ex)' which binds just the free occurrences of 'x' in its scope. Which quantifiers bind which occurrences of 'x' can be indicated with connecting lines in this way:

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(Ex)[(x)Lxa -> Lxb]
   1   2 3      4
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The quantifier at 1 binds the 'x' at 4, and the quantifier at 2 binds the 'x' at 3.

Here are the definitions which make all of this precise:

The SCOPE of a quantifier is the shortest full following sentences, as indicated by parentheses.

A variable, u is FREE just in case it does not occur in the scope of any quantifier (u) or (Eu)

A quantifier (u) or (Eu) BINDS all and only the free occurrences of u in it scope. These quantifiers do not bind an occurrence of u in their scope if these occurrences are already bound by some other quantifier.

Exercises 3-1. Draw lines showing which quantifiers bind which variables. Also specify the bound and free occurrences of the variables, using the numbers.

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a) Lzz,     b) (y)(z)Lzy
    12                12

c) (z)(Bz -> Lxz)    d) (Ex)[Lxz & (y)(Lxy v Lzx)]
     1         23              12          34    56
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I marked some of the definitions in the last chapter as 'incomplete'. This was because they did not take into account the complications of multiple quantification and of the occurrence of free variables. With the notions of scope of a quantifier and of free and bound variables we can now straighten this out.

> The SUBSTITUTION INSTANCE of (u)(...u...), with the name s substituted for u, is (...s...), the sentence which results when you drop the initial (u) and write s for all FREE occurrences of u in (...u...).
>
> The SUBSTITUTION INSTANCE of (Eu)(...u...), with the name s substituted for u, is (...s...), the sentence which results when you drop the initial (Eu) and write s for all FREE occurrences of u in (...u...).
>
> If a sentence has one or more free variables it is said to be an OPEN SENTENCE. A sentences with no free variables is called a CLOSED SENTENCE.

The definitions of truth for a quantified sentence do not work for open sentences. Open sentence do not get assigned truth values in interpretations and so, in a sense, are not quite proper sentences. Thus we say:

> A universally quantifieed closed sentences is true in an interpretation if and only if ALL of the sentence's substitution instances, formed with the names in the interpretation, are true in the interpretation.
>
> An existentially quantifieed closed sentences is true in an interpretation if and only if SOME ~~all~~ (that is, one or more) of the sentence's substitution instances, formed with the names in the interpretation, are true in the interpretation.

Exercises 3-2. Provide subsitution instances for each of the following sentences, using the name 'a': a) (y)(Ex)Lxy b) (Ez)[(x)Bx v Bz] c) (Ex)[Bx <-> (x)(Lax v Bx)].